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Regular Cyclic Coverings of the Platonic Maps

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We use homological methods to describe the regular maps and hypermaps which are cyclic coverings of the Platonic maps, branched over the face centers, vertices or midpoints of edges.

1. INTRODUCTION

The Möbius–Kantor map $\{4+4, 3\}$ [6, Sections 8.8 and 8.9] is a regular orientable map of type $\{8, 3\}$ and genus 2. It is a 2-sheeted covering of the cube $\{4, 3\}$, branched over the centers of its six faces, each of which lifts to an octagonal face. Its (orientation-preserving) automorphism group is isomorphic to $GL_2(3)$, a double covering of the automorphism group $PGL_2(3) \cong S_4$ of the cube. The aim of this note is to describe all the regular maps and hypermaps which can be obtained in a similar manner as cyclic branched coverings of the Platonic maps $\mathcal{M}$, with the branching at the face centers, vertices, or midpoints of edges. The method used is to consider the action of $\mathrm{Aut}\,\mathcal{M}$ on certain homology modules; in a companion paper [16] we use cohomological techniques to give explicit constructions of these coverings in terms of voltage assignments. Conder and Everitt [2] have used different methods to construct non-orientable regular maps as cyclic coverings of smaller maps, branched over their face centers.

Let $\mathcal{M}$ be a Platonic map, that is, a regular map on the sphere S^2. In the notation of [6], $\mathcal{M}$ has type $\{n, m\}$, or simply $\mathcal{M} = \{n, m\}$, where the faces are n-gons and the vertices have valency m; as a hypermap, $\mathcal{M}$ has type $(m, 2, n)$. Here, $0 \leq (m-2)(n-2) < 4$, so $\mathcal{M}$ is the dihedron $\{n, 2\}$, the hosohedron $\{2, m\}$, the tetrahedron $\{3, 3\}$, the cube $\{4, 3\}$, the octahedron $\{3, 4\}$, the dodecahedron $\{5, 3\}$ or the icosahedron $\{3, 5\}$.

We first determine the regular maps $\mathcal{N}$ which are d-sheeted coverings of $\mathcal{M}$, with a cyclic group of covering transformations, branched over the face centers. Thus $\mathcal{N}$ is a map of type $\{dn, m\}$, or equivalently, a hypermap of type $(m, 2, dn)$, and the group $\tilde{G} := \mathrm{Aut}\,\mathcal{N}$ has a normal subgroup $D \cong C_d$ with $\mathcal{N}/D \cong \mathcal{M}$, so that $\tilde{G}/D \cong G := \mathrm{Aut}\,\mathcal{M}$. Our main result is:

THEOREM 1. *The isomorphism classes of d-sheeted regular cyclic coverings $\mathcal{N}$ of $\mathcal{M}$, branched over the face centers, are in one-to-one correspondence with the solutions $u \in \mathbf{Z}_d$ of*

$$u^h = 1 \qquad \text{and} \qquad 1 + u + u^2 + \cdots + u^{f-1} = 0, \tag{$*$}$$

where $h = \mathrm{hcf}\,(m, 2)$ and $\mathcal{M}$ has f faces. They have type $\{dn, m\}$ and genus $(d-1)(f-2)/2$, and are all reflexible. The group $\tilde{G} = \mathrm{Aut}\,\mathcal{N}$ has a presentation

$$\langle x, y, z \mid x^m = y^2 = z^{dn} = xyz = 1,\ (z^n)^x = z^{nu} \rangle,$$

with $G = \mathrm{Aut}\,\mathcal{M} \cong \tilde{G}/D$ where $D = \langle z^n \rangle$.

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When $u = 1$ (equivalently, D is in the center of $\tilde{G}$), we obtain Sherk's maps $\{d \cdot n, m\}$ [14], one for each d dividing f; if m is odd these are the only possibilities, but if m is even we also obtain non-central cyclic coverings for certain values of d, including $d = \infty$ in some cases. By duality, a similar process yields those d-sheeted regular cyclic covers $\mathcal{N}$ of $\mathcal{M}$ which are branched over the vertices; these are maps of type $\{n, dm\}$. Finally, if we allow branching over the edges of $\mathcal{M}$ we obtain d-sheeted coverings $\mathcal{N}$ which are hypermaps (but not maps) of type $(m, 2d, n)$, the relevant conditions being $u^h = 1,\ 1 + u + u^2 + \cdots + u^{e-1} = 0$ where $h = \mathrm{hcf}(m, n)$ and $\mathcal{M}$ has e edges; $\mathcal{N}$ is reflexible except for the case $\mathcal{M} = \{3, 3\}$ with $u \neq 1$.

2. Preliminaries

First we briefly sketch the connections between maps, hypermaps (always assumed to be orientable) and triangle groups; for the details, see [7–9], and for background on hypermaps, see [5]. We define a triangle group to be

$$\Delta(p, q, r) = \langle x, y, z \mid x^p = y^q = z^r = xyz = 1\rangle,$$

where $p, q, r \in \mathbf{N} \cup \{\infty\}$ and we ignore any relation $g^\infty = 1$. Any m-valent map $\mathcal{N}$ corresponds to a subgroup N of the triangle group

$$\Delta := \Delta(m, 2, \infty) = \langle x, y, z \mid x^m = y^2 = xyz = 1\rangle,$$

with vertices, edges and faces corresponding to the cycles of x, y and z on the cosets of N. The map $\mathcal{N}$ is regular if and only if N is normal in Δ, in which case $\mathrm{Aut}\,\mathcal{N} \cong \Delta/N$. In particular, the Platonic map $\mathcal{M} = \{n, m\}$ corresponds to the normal closure M of z^n in Δ, and $\mathrm{Aut}\,\mathcal{M} \cong \Delta/M \cong \Delta(m, 2, n)$. The regular map $\mathcal{N}$ is a d-sheeted covering of $\mathcal{M}$ if and only if N is a subgroup of index d in M, in which case the group of covering transformations is M/N. The regular m-valent maps which are d-sheeted cyclic coverings of $\mathcal{M}$ are therefore in bijective correspondence with the subgroups N of M which are normal in Δ, with $M/N \cong C_d$. Since M/N is abelian and has exponent d, such subgroups N contain the commutator subgroup M' and the subgroup M^d generated by the dth powers in M, so they correspond to subgroups $\overline{N} = N/M'M^d$ of $\overline{M} = M/M'M^d$. The action of Δ by conjugation on the normal subgroup M preserves its characteristic subgroups M' and M^d, so there is an induced action of Δ on $\overline{M}$; since M is in the kernel of this action, we therefore have an action of the group $G = \mathrm{Aut}\,\mathcal{M} \cong \Delta/M$ on $\overline{M}$, which is a module for G over the ring $\mathbf{Z}_d$; it follows that a subgroup N of M, containing $M'M^d$, is normal in Δ if and only if $\overline{N}$ is a G-invariant submodule of $\overline{M}$.

Let $S = S^2 \setminus \{c_1, \ldots, c_f\}$, where $c_1, \ldots, c_f$ are the centers of the f faces of $\mathcal{M}$. Then M can be identified with the fundamental group $\pi_1(S)$ of S, a free group of rank $f - 1$ generated by the homotopy classes g_i of loops around the punctures c_i, with a single defining relation $g_1 \ldots g_f = 1$. It follows that the group $M^{\mathrm{ab}} = M/M'$ can be identified with the first integer homology group $H_1(S; \mathbf{Z}) = \pi_1(S)^{\mathrm{ab}}$ of S, a free abelian group of rank $f - 1$ generated by the homology classes $[g_i]$ with $[g_1] + \cdots + [g_f] = 0$, and then by the universal coefficient theorem, $\overline{M}$ is identified with the mod (d) homology group $H_1(S; \mathbf{Z}_d) = H_1(S; \mathbf{Z}) \otimes \mathbf{Z}_d \cong \mathbf{Z}_d^{f-1}$. Under these identifications, the actions of G induced by conjugation in Δ and by homeomorphisms of S are the same, so our problem is to find the G-submodules of $H_1(S; \mathbf{Z}_d)$ of codimension 1, or equivalently, the kernels of G-epimorphisms onto one-dimensional G-modules.

Regular abelian coverings of maps (and more generally Riemann surfaces) with automorphism group G can be determined by using ordinary and modular representation theory to study the decomposition of the G-module $H_1(S)$ over various rings and fields of coefficients;

see [10, 11, 13] for examples of this technique. In our case, since we are interested only in one-dimensional constituents, we can adopt a rather simpler, more direct approach.

Let P be the permutation module over $\mathbf{Z}_d$ for the action of G on the faces of $\mathcal{M}$. As a $\mathbf{Z}_d$-module, this has a basis $e_1, \ldots, e_f$ in one-to-one correspondence with the faces, and these are permuted in the same way as G permutes the faces. Now $e_1 + \cdots + e_f$ generates a G-invariant one-dimensional submodule P_1 of P, and $H_1(S; \mathbf{Z}_d)$ is isomorphic to the quotient G-module P/P_1. We therefore need to find the G-submodules of codimension 1 in P containing P_1, or equivalently, the G-epimorphisms $\theta : P \to Q$ where Q is one-dimensional and $P_1 \leq \ker\theta$.

3. Central Cyclic Coverings

We first consider the case where D is central in $\tilde{G}$, so that G acts trivially on Q. The resulting maps $\mathcal{N}$ were described by Sherk [14] from a rather different point of view, but for completeness we will show how they arise from the above general theory. For notational convenience, we let $\mathbf{Z}_\infty$ denote $\mathbf{Z}$.

Lemma 2. *Let G be a transitive permutation group of degree f, let P be its permutation module over $\mathbf{Z}_d$ $(d \in \mathbf{N} \cup \{\infty\})$, with basis $e_1, \ldots, e_f$ permuted by G, let P_1 be the one-dimensional G-submodule of P spanned by $\sum e_i$, and let Q be a one-dimensional G-module with the trivial action of G. Then there is a G-epimorphism $\theta : P \to Q$ with $\ker\theta \geq P_1$ if and only if d is finite and divides f, in which case $\ker\theta$ is the G-submodule $P^1 = \{\sum a_i e_i \mid \sum a_i = 0\}$.*

Proof. As a $\mathbf{Z}_d$-module, P is free of rank f, so homomorphisms $\theta : P \to Q$ of $\mathbf{Z}_d$-modules correspond bijectively to choices of elements $q_i = e_i\theta \in Q$; since G acts transitively on the elements e_i and trivially on the elements q_i, such a group homomorphism θ is a homomorphism of G-modules if and only if $q_1 = \cdots = q_f$, say $q_i = q$ for all i. Such a homomorphism θ is an epimorphism if and only if q generates the additive group $Q \cong \mathbf{Z}_d$, that is, if and only if q is a unit in $\mathbf{Z}_d$. Now $P_1 \leq \ker\theta$ if and only if $\sum q_i = 0$, or equivalently, $fq = 0$ in $\mathbf{Z}_d$. Since q is a unit, this is equivalent to $f = 0$ in $\mathbf{Z}_d$, that is, d is finite and divides f in $\mathbf{Z}$. An element $\sum a_i e_i \in P$ is in $\ker\theta$ if and only if $\sum a_i q = 0$, or equivalently, $\sum a_i = 0$, so $\ker\theta = P^1$. □

Applying this result to the G-module $\overline{M} \cong H_1(S; \mathbf{Z}_d) \cong P/P_1$, we obtain a unique subgroup $N \leq M$ for each divisor $d = |M : N|$ of f, and no other subgroups, so the number of coverings of $\mathcal{M}$ we obtain is equal to the number $\tau(f)$ of divisors of f. For each divisor d, the resulting map $\mathcal{N}$ has type $\{dn, m\}$, so following Sherk [14] we will denote it by $\{d \cdot n, m\}$. Since the covering $\mathcal{N} \to \mathcal{M}$ has d sheets, and there are f branch points of order $d-1$, the Riemann–Hurwitz formula implies that $\mathcal{N}$ has genus

$$g = 1 - d + \frac{1}{2}f(d-1) = \frac{1}{2}(d-1)(f-2).$$

Any regular map (or hypermap) $\mathcal{N}$ is *reflexible* if it is isomorphic to its mirror image $\overline{\mathcal{N}}$, or equivalently, if the corresponding normal subgroup N of the triangle group Δ is also normal in the extended triangle group which contains Δ with index 2; otherwise, $\mathcal{N}$ and $\overline{\mathcal{N}}$ form a *chiral pair*, non-isomorphic but with the same type, genus and automorphism group. It is well known that the Platonic maps $\mathcal{M}$ are all reflexible, and by the uniqueness of the covering of $\mathcal{M}$ for each d, it immediately follows that each map $\{d \cdot n, m\}$ is reflexible.

PROPOSITION 3. *The group* $\tilde{G} = \mathrm{Aut}\,\mathcal{N}$ *has a presentation*

$$\langle x, y, z \mid x^m = y^2 = z^{dn} = xyz = [x, z^n] = 1\rangle.$$

PROOF. Let H denote the group defined by this presentation. As the automorphism group of a regular map of type $\{dn, m\}$ with z^n central, $\tilde{G}$ satisfies all the relations in this presentation and is therefore an epimorphic image of H. However, the relations of H imply that z^n commutes with x and z, which generate H, so H has a central subgroup $\langle z^n\rangle$ of order dividing d, with $H/\langle z^n\rangle \cong \Delta(m, 2, n) \cong G$. Thus $|H| \leq d|G| = |\tilde{G}|$, so $\tilde{G} \cong H$. □

We will use the *ATLAS* [4] notation $d.G$ for a group, such as $\tilde{G}$, with a cyclic normal subgroup of order d, with quotient group G. In general, this notation does not specify the group uniquely, but given that we are considering cyclic central regular coverings, branched over the faces, the preceding arguments show that $\mathcal{M}$ and d determine $\mathcal{N}$ and hence $\tilde{G}$ uniquely, so for our purposes this notation is (at least locally) unambiguous. In later sections, however, we will consider branching over vertices or edges, so the notation $d.G$ may then describe a different group with this structure. The following result tells us when D is a direct factor of $\tilde{G}$, so that $\tilde{G} \cong D \times G$.

PROPOSITION 4. *The following are equivalent:*

(1) *D is a direct factor of $\tilde{G}$; and*
(2) *d is coprime to n, and either* (a) *d divides m, or* (b) *$d \equiv 2 \bmod (4)$ and d divides $2m$.*

PROOF. Since D is normal in $\tilde{G}$, it is a direct factor if and only if there is a homomorphism $\phi : \tilde{G} \to D$ which restricts to the identity on D, so that $\tilde{G} = D \times \ker\phi$. Since $D = \langle z^n\rangle$ we require $z\phi = z^{ni}$ for some i, so ϕ is the identity on D if and only if $z^n = z^n\phi = z^{n^2 i}$; this is equivalent to dn dividing $n - n^2 i$, that is, d dividing $1 - ni$, which is possible if and only if n (and i) are coprime to d. Since $y^2 = 1$ in $\tilde{G}$ we require $(y\phi)^2 = 1$ in D, so either (a) $y\phi = 1$ or (b) $y\phi = (z^n)^{d/2}$ where d is even. In case (a) the relation $xyz = 1$ forces $x\phi = z^{-ni}$, so we have a homomorphism ϕ if and only if $(z^{-ni})^m = 1$, that is, d divides m. In case (b) we have $x\phi = z^{-n(i+d/2)}$, so we require d to divide $m(2i + d)/2$; now d and $(2i + d)/2$ have highest common factor 1 or 2 as $d \equiv 0$ or $2 \bmod (4)$, so we require m to be divisible by d or $d/2$, respectively. □

More generally, a similar calculation (which we will omit) can be performed for *any* divisor d of f, by considering a homomorphism ϕ from $\tilde{G}$ onto a cyclic group C_e generated by the image of z^n. This shows that $\tilde{G}$ is a central product $D \circ E$ of D and a normal subgroup $E = \ker\phi$ of index e, for any e dividing $\mathrm{hcf}(m, d)/\mathrm{hcf}(m, d, n)$; specifically, $\tilde{G}$ is isomorphic to the quotient $D \circ_C E$ of $D \times E$ formed by identifying the unique subgroup C of index e in D with an isomorphic subgroup in the center of E, so that $D \cap E = C$. If m and n are odd, we also obtain such a decomposition of $\tilde{G}$ for even e dividing $\mathrm{hcf}(2m, d)/\mathrm{hcf}(2m, d, n)$, by taking $y\phi \neq 1$. Proposition 4 describes the circumstances in which $e = d$, so that $C = 1$ and $\tilde{G} = D \times E$ with $E \cong G$. In each specific case, a presentation for E can be obtained from that of $\tilde{G}$ by the Reidemeister–Schreier method, thus enabling E to be identified.

For each $\mathcal{M} = \{n, m\}$, the largest covering $\{d \cdot n, m\}$ obtained is $\mathcal{N} = \{f \cdot n, m\}$. Its automorphism group $f.G$ has a presentation

$$\langle x, y, z \mid x^m = y^2 = z^{fn} = xyz = [x, z^n] = 1\rangle$$

which can, in fact, be simplified to

$$\langle x, y, z \mid x^m = y^2 = xyz = [x, z^n] = 1\rangle.$$

To see this, let H denote the group with this second presentation, so there is a natural epimorphism $H \to f.G$. Clearly H is a quotient of $\Delta(m, 2, \infty)$, and has a central subgroup $\langle z^n \rangle$ with quotient $\Delta(m, 2, n) \cong G$, so it corresponds to a regular m-valent map $\mathcal{N}$ which is a cyclic central cover of $\mathcal{M}$, branched over the faces. Lemma 2 shows that the degree d of such a covering must be finite and must divide f, so $|H| \leq f|G| = |f.G|$ and hence $f.G \cong H$.

If we put $r = y^{-1}$ $(= y)$ and $s = x^{-1}$ we see that $f.G$ has a presentation

$$\langle r, s \mid r^2 = s^m = 1,\ (rs)^n = (sr)^n \rangle;$$

this is because $(rs)^n = (sr)^n$ if and only if $(rs)^n$ commutes with s, that is, z^n commutes with x. Thus $f.G$ is the special case $l = 2$ of the group

$$\langle r, s \mid r^l = s^m = 1,\ (rs)^n = (sr)^n \rangle$$

denoted by $l\,[2n]m$ in [6, Section 6.7], where it is also shown that z^n has order $|G|/n = f$.

The automorphism groups $d.G$ of the other coverings $\{d \cdot n, m\}$ of $\mathcal{M}$ can be obtained from $f.G$ by factoring out the unique subgroup $\langle z^{dn} \rangle$ of index d in $\langle z^n \rangle$; these coverings form a lattice isomorphic to the lattice of divisors d of f.

4. The Cyclic Central Covering Maps

We now consider the Platonic maps $\mathcal{M}$ in turn, briefly describing the cyclic central coverings $\mathcal{N}$ which arise in each case, together with their automorphism groups. (See [6, Sections 4.2 and 5.1] for the Platonic maps and groups, and [14, Table I] for a list of the maps $\mathcal{N}$.)

(a) When $\mathcal{M}$ is the cube $\{4, 3\}$, so that $G \cong S_4 \cong PGL_2(3)$, we have $m = 3$, $n = 4$ and $f = 6$. The divisors of f are $d = 1, 2, 3, 6$, giving regular maps $\mathcal{N} = \{d \cdot 4, 3\}$ of genus $g = 2(d-1) = 0, 2, 4, 10$ and type $\{4d, 3\} = \{4, 3\},\ \{8, 3\},\ \{12, 3\},\ \{24, 3\}$. The first map is the cube $\mathcal{M} = \{4, 3\}$, and the second is the Möbius–Kantor map, denoted by $\{4 + 4, 3\}$ in [6, Section 8.8]. The automorphism group of $\mathcal{N}$, of the form $d.S_4$, has order $24d$ and is given by

$$\mathrm{Aut}\,\mathcal{N} = \langle x, y, z \mid x^3 = y^2 = z^{4d} = xyz = [x, z^4] = 1 \rangle.$$

By Proposition 4, this is isomorphic to $C_d \times S_4$ if and only if $d = 1$ or 3. If $d = 2$ then $\mathrm{Aut}\,\mathcal{N} \cong GL_2(3)$, a non-split double covering of G, with x, y and z corresponding to the matrices $\binom{1\ 0}{1\ 1}$, $\binom{2\ 1}{0\ 1}$ and $\binom{1\ 1}{2\ 1}$; this group is denoted by $\langle -3, 4 \mid 2 \rangle$ in [6, Section 6.6 and Table 9]. If $d = 6$ then $\mathrm{Aut}\,\mathcal{N}$ is a central product $D \circ_C E \cong C_6 \circ_{C_2} GL_2(3) \cong C_3 \times GL_2(3)$.

(b) When $\mathcal{M}$ is the tetrahedron $\{3, 3\}$, so that $G \cong A_4 \cong PSL_2(3)$, we have $m = 3$, $n = 3$ and $f = 4$, so $d = 1, 2, 4$, giving regular maps $\mathcal{N} = \{d \cdot 3, 3\}$ of genus $g = d - 1 = 0, 1, 3$ and type $\{3, 3\}, \{6, 3\}, \{12, 3\}$, with automorphism groups $d.A_4$ of order $12d$ given by

$$\mathrm{Aut}\,\mathcal{N} = \langle x, y, z \mid x^3 = y^2 = z^{3d} = xyz = [x, z^3] = 1 \rangle.$$

When $d = 1$ we have the tetrahedron, while $\{2 \cdot 3, 3\}$ is the torus map $\{6, 3\}_{2,0}$ discussed in Section 8.4 of [6], with $\mathrm{Aut}\,\mathcal{N} \cong C_2 \times A_4$ by Proposition 4. If $d = 4$ then $\mathrm{Aut}\,\mathcal{N}$ is a central product $D \circ_C E$, where E is the binary tetrahedral group $\langle 3, 2, 3 \rangle \cong SL_2(3)$ (a double covering of G, see [6, Section 6.5]), and $C \cong C_2$; one can identify D with the group of matrices $\lambda I \in GL_2(3^2)$ satisfying $\lambda^4 = 1$, and E with the subgroup $SL_2(3) \leq GL_2(3^2)$, so that $\mathrm{Aut}\,\mathcal{N} = DE \leq GL_2(3^2)$.

(c) If $\mathcal{M}$ is the octahedron $\{3, 4\}$, so that $G \cong S_4 \cong PGL_2(3)$ as in (a), we have $m = 4$, $n = 3$ and $f = 8$. Thus $d = 1, 2, 4, 8$ and we have regular maps $\mathcal{N} = \{d \cdot 3, 4\}$ of genus

$g = 3(d-1) = 0, 3, 9, 21$ and type $\{3, 4\}, \{6, 4\}, \{12, 4\}, \{24, 4\}$. They have automorphism groups $d.S_4$ of order $24d$ given by

$$\mathrm{Aut}\,\mathcal{N} = \langle x, y, z \mid x^4 = y^2 = z^{3d} = xyz = [x, z^3] = 1\rangle,$$

isomorphic $C_d \times S_4$ for $d = 1, 2, 4$. (Thus $2.S_4$ denotes $C_2 \times S_4$ here, whereas in (a) it denotes $GL_2(3)$.) If $d = 8$ then $\mathrm{Aut}\,\mathcal{N} = D \circ_C E \cong C_8 \circ_{C_2} GL_2(3)$; one can identify D with the center of $GL_2(3^2)$, consisting of the scalar matrices λI $(\lambda \neq 0)$, and E with $GL_2(3)$, so that $\mathrm{Aut}\,\mathcal{N} = DE \leq GL_2(3^2)$.

(d) When $\mathcal{M}$ is the dodecahedron $\{5, 3\}$ we have $G \cong A_5 \cong PSL_2(5)$, with $m = 3,\ n = 5$ and $f = 12$. Thus $d = 1, 2, 3, 4, 6, 12$, giving regular maps $\mathcal{N} = \{d \cdot 5, 3\}$ of genus $g = 5(d-1) = 0, 5, 10, 15, 25, 55$ and type $\{5, 3\}, \{10, 3\}, \{15, 3\}, \{20, 3\}, \{30, 3\}, \{60, 3\}$. Each $\mathcal{N}$ has automorphism group $d.A_5$ of order $60d$ given by

$$\mathrm{Aut}\,\mathcal{N} = \langle x, y, z \mid x^3 = y^2 = z^{5d} = xyz = [x, z^5] = 1\rangle.$$

By Proposition 4, this is isomorphic to $C_d \times A_5$ for $d = 1, 2, 3, 6$. When $d = 4$ or 12, however, $\mathrm{Aut}\,\mathcal{N}$ is the central product of $D \cong C_d$ and $E \cong SL_2(5)$ (the binary icosahedral group), amalgamating a subgroup C_2; by representing D as a group of scalar matrices, as in the preceding examples, one can embed this group in $GL_2(5)$ or $GL_2(5^2)$, respectively.

(e) If $\mathcal{M}$ is the icosahedron $\{3, 5\}$ we again have $G \cong A_5 \cong PSL_2(5)$, but now $m = 5,\ n = 3$ and $f = 20$. Thus $d = 1, 2, 4, 5, 10, 20$, so we obtain regular maps $\mathcal{N} = \{d \cdot 3, 5\}$ of genus $g = 9(d-1) = 0, 9, 27, 36, 81, 171$ and type $\{3, 5\}, \{6, 5\}, \{12, 5\}, \{15, 5\}, \{30, 5\}, \{60, 5\}$, with automorphism groups $d.A_5$ of order $60d$ given by

$$\mathrm{Aut}\,\mathcal{N} = \langle x, y, z \mid x^5 = y^2 = z^{3d} = xyz = [x, z^3] = 1\rangle.$$

This is isomorphic to $C_d \times A_5$ for $d = 1, 2, 5, 10$. However, for $d = 4, 20$ it is a central product $D \circ_C E \cong C_d \circ_{C_2} SL_2(5)$; when $d = 4$ this can be embedded in $GL_2(5)$, and when $d = 20$ it has the form $C_5 \times (C_4 \circ_{C_2} SL_2(5))$.

(f) If $\mathcal{M}$ is the dihedron $\{n, 2\}$, with two n-gonal faces separated by a circuit of n vertices and n edges, then $G \cong D_n$, the dihedral group of order $2n$. Here $m = 2$ and $f = 2$, so $d = 1$ or 2, corresponding to the maps $\mathcal{N} = \{n, 2\}$ and $\{2 \cdot n, 2\} = \{2n, 2\}$ of genus 0. The second map has

$$\mathrm{Aut}\,\mathcal{N} = \langle x, y, z \mid x^2 = y^2 = z^{2n} = xyz = [x, z^n] = 1\rangle \cong D_{2n}.$$

When n is odd this group has the form $D \times E \cong C_2 \times D_n$, but when n is even the central extension does not split.

(g) Let $\mathcal{M}$ be the hosohedron $\{2, m\}$, the dual of $\{m, 2\}$ with two vertices and m 2-gonal faces. Then $G \cong D_m$ with $n = 2$ and $f = m$, so the relevant values of d are the divisors of m. The resulting maps $\mathcal{N} = \{d \cdot 2, m\}$ have type $\{2d, m\}$ and genus $(d-1)(m-2)/2$, with automorphism groups $d.D_m$ of order $2dm$ given by

$$\mathrm{Aut}\,\mathcal{N} = \langle x, y, z \mid x^m = y^2 = z^{2d} = xyz = [x, z^2] = 1\rangle.$$

When d is odd this has the form $D \times E \cong C_d \times D_m$, by Proposition 4. Taking $d = 2$ we obtain the well-known result that for each integer $g \geq 0$ there is a regular map of genus g with $8(g+1)$ automorphisms (see [1, 12] for the analogous result for Riemann surfaces).

5. Riemann Surfaces and Galois Groups

Here we introduce a digression, showing how the central coverings considered above can be viewed in terms of Riemann surfaces, algebraic curves and Galois groups. This section is not essential for the rest of the paper.

If we identify S^2 with the Riemann sphere $\Sigma = \mathbf{C} \cup \{\infty\}$ by stereographic projection, then we can use the covering $\mathcal{N} \to \Sigma$ to impose a complex structure on $\mathcal{N}$, which therefore lies on a compact Riemann surface X of genus g. This surface can be represented as an algebraic curve by the equation $w^d = \prod_i'(z - c_i)$, where $\prod'$ denotes the product as c_i ranges over the face centers of $\mathcal{M}$ in $\mathbf{C}$ (excluding ∞), and the covering $\mathcal{N} \to \mathcal{M}$ is induced by the projection $X \to \Sigma,\ (z, w) \mapsto z$. The group $\tilde{G}$ acts as a group of conformal automorphisms of X, and the subgroup D consists of the automorphisms $(z, w) \mapsto (z, \zeta w)$ where $\zeta^d = 1$.

In case (a) of Section 4, for instance, if we take the face centers of the cube $\mathcal{M}$ to be the points ± 1, $\pm i$, 0 and ∞ in Σ, we obtain the smooth affine curve $w^d = z^5 - z$ (see [6, Section 8.8]). In case (f), where $\mathcal{M} = \{n, 2\}$, we have $X = \Sigma$; if we take the face centers of $\mathcal{M}$ and $\mathcal{N}$ to be 0 and ∞, and their vertices to be the nth and $2n$th roots of 1, then the covering $\mathcal{N} \to \mathcal{M}$ is given by $w \mapsto w^2$. In case (g), by taking the face centers of $\mathcal{M} = \{2, m\}$ to be the mth roots of 1 (with vertices at 0 and ∞), we obtain the curve $w^d = z^m - 1$, conformally equivalent to the generalized Fermat curve $w^d + z^m = 1$; when $d = 2$ this is the Accola–Maclachlan curve [1, 12], and when $d = m$ it is the mth degree Fermat curve.

To see this connection more precisely, consider the homogeneous polynomial

$$F(x_0, x_1, x_2) = x_1^d x_2^{f-d} - \prod_{i=1}^{f} L_i(x_0, x_2) \in \mathbf{C}[x_0, x_1, x_2],$$

where

$$L_i(x_0, x_2) = \begin{cases} x_0 - c_i x_2 & \text{if } c_i \in \mathbf{C}, \\ x_2 & \text{if } c_i = \infty. \end{cases}$$

The zero set

$$V(F) = \{\, [x_0, x_1, x_2] \in \mathbf{P}^2(\mathbf{C}) \mid F(x_0, x_1, x_2) = 0 \,\}$$

is an algebraic curve birationally equivalent to X, whose only singularities are 'at infinity' (where $x_2 = 0$). The affine part of this curve, given by $x_2 \neq 0$, is the set

$$X_0 = \left\{ [z, w, 1] \in \mathbf{P}^2(\mathbf{C}) \mid w^d = \prod_i{}'(z - c_i) \right\},$$

where $z = x_0/x_2$ and $w = x_1/x_2$, and the restriction of the covering $X \to \Sigma$ to X_0 is simply $[z, w, 1] \mapsto z$. The group $\tilde{G}$ acts as a group of conformal automorphisms of X, and the central subgroup D consists of the automorphisms $[x_0, x_1, x_2] \mapsto [x_0, \zeta x_1, x_2]$ where $\zeta^d = 1$. In the extreme case, when $d = f$, $V(F)$ is everywhere smooth; then X can be represented as the smooth projective plane curve $V(F) \subseteq \mathbf{P}^2(\mathbf{C})$, and the covering map is the meromorphic function $z : [x_0, x_1, x_2] \mapsto x_0/x_2$.

Equivalently, the above picture can be viewed in terms of Galois theory, as follows. If we let $\mathrm{Mer}\,(X)$ denote the field of meromorphic functions on a Riemann surface X, then the assignment $X \mapsto \mathrm{Mer}\,(X)$ is functorial; in particular, the group of conformal automorphisms of X corresponds to the Galois group of $\mathrm{Mer}\,(X)$ over its subfield $\mathbf{C}$ of constant functions. Any meromorphic function $X \to Y$ induces a field extension $\mathrm{Mer}\,(Y) \hookrightarrow \mathrm{Mer}\,(X)$. This covering is regular if every conformal automorphism of Y can be lifted to one of X, and this happens precisely when $\mathrm{Mer}\,(X)$ is a normal (and hence Galois) extension of $\mathrm{Mer}\,(Y)$. In the cases considered above, X is a Riemann surface with $\mathrm{Mer}\,(X) = \mathbf{C}(w, z)$ where $w^d = \prod_i'(z - c_i)$. The fact that $z : X \to \Sigma$ is a central cyclic branched covering of degree d is mirrored by the fact that $\mathbf{C}(z) \subseteq \mathrm{Mer}\,(X)$ is a normal extension with Galois group $D \cong C_d$ (induced by $w \mapsto \zeta w$ where $\zeta^d = 1$); furthermore, $\mathrm{Mer}\,(X)$ is a normal extension of the fixed field F of $\tilde{G}$ in $\mathrm{Mer}\,(X)$, with Galois group $\tilde{G}$, and $G = \tilde{G}/D$ is the Galois group of the (normal) extension $F \subseteq \mathbf{C}(z)$.

6. Branching Over Vertices

One can also consider cyclic central coverings $\mathcal{N}$ of Platonic maps $\mathcal{M} = \{n, m\}$, where the branching is over the vertices rather than the faces of $\mathcal{M}$. However, the dual map $\mathcal{N}'$ is then a cyclic central covering of the Platonic map $\mathcal{M}' = \{m, n\}$, branched over the faces of $\mathcal{M}'$, so $\mathcal{N}$ is simply the dual map (Sherk's $\{n, d \cdot m\}$) of $\{d \cdot m, n\}$, with the same genus g and automorphism group $\tilde{G}$. For instance, by branching over the vertices of a cube (equivalently, over the faces of an octahedron and then dualizing) we obtain d-sheeted covering maps $\{4, 3 \cdot d\}$ of genus $3(d-1)$ for $d = 1, 2, 4$ and 8.

7. Branching Over Edges

If we allow hypermaps, rather than just maps, then by branching over the midpoints of the edges of $\mathcal{M}$ we obtain regular hypermaps $\mathcal{N}$ of type $(m, 2d, n)$ as d-sheeted cyclic central coverings of $\mathcal{M}$. Arguments similar to those used before, but now with $\Delta = \Delta(m, \infty, n)$, show that there is one such covering $\mathcal{N}$ for each divisor d of e, where $e = |G|/2$ is the number of edges of $\mathcal{M}$. This hypermap is reflexible, of genus $(d-1)(e-2)/2$, and the equation of the underlying algebraic curve is $w^d = \prod_i'(z - c_i)$ where the product is over the edge centers $c_i \neq \infty$ of $\mathcal{M}$. The automorphism group $\tilde{G}$ of $\mathcal{N}$ has a presentation

$$\operatorname{Aut}\mathcal{N} = \langle x, y, z \mid x^m = y^{2d} = z^n = xyz = [x, y^2] = 1\rangle;$$

this group has the form $d.G$, having a cyclic central subgroup $\langle y^2\rangle$ of order d with quotient group isomorphic to G. The largest covering is that for $d = e$, with

$$\begin{aligned}\operatorname{Aut}\mathcal{N} &= \langle x, y, z \mid x^m = y^{2e} = z^n = xyz = [x, y^2] = 1\rangle \\ &= \langle x, y, z \mid x^m = z^n = xyz = [x, y^2] = 1\rangle\end{aligned}$$

isomorphic to the group

$$m\,[4]\,n = \langle r, s \mid r^m = s^n = 1,\ (rs)^2 = (sr)^2\rangle$$

discussed in Section 6.7 of [6].

8. Non-central Cyclic Coverings

We now remove the restriction that D should be central in $\tilde{G}$, so that G need not act trivially on the module Q. If we identify D and Q with $\mathbf{Z}_d$, then the action of G on both is given by a homomorphism α from G to $\operatorname{Aut}\mathbf{Z}_d = U_d$, the group of units mod (d), with each $g \in G$ inducing multiplication by the unit $g\alpha$. The most important content of our main theorem is provided by the following result. Recall that $h = \operatorname{hcf}(m, 2)$ where $\mathcal{M} = \{n, m\}$, and $\mathcal{M}$ has f faces.

Proposition 5. *The d-sheeted regular cyclic coverings $\mathcal{N}$ of $\mathcal{M}$, branched over the face centers, are in one-to-one correspondence with the solutions $u \in \mathbf{Z}_d$ of*

$$u^h = 1, \qquad 1 + u + u^2 + \cdots + u^{f-1} = 0. \tag{$*$}$$

The group $\tilde{G} = \operatorname{Aut}\mathcal{N}$ has a presentation $\langle x, y, z \mid x^m = y^2 = z^{dn} = xyz = 1,\ (z^n)^x = z^{nu}\rangle$, with $\operatorname{Aut}\mathcal{M} \cong \tilde{G}/D$ where $D = \langle z^n\rangle$.

PROOF. As before, homomorphisms $\theta : P \to Q$ of $\mathbf{Z}_d$-modules correspond to choices of elements $q_i = e_i\theta \in Q$. For θ to be a G-module homomorphism, we require that $(e_i g)\theta = (e_i\theta)g$ for all i and all $g \in G$, that is, if $e_i g = e_j$ then $q_j = q_i(g\alpha)$ in $\mathbf{Z}_d$. Since G acts transitively on the elements e_i, it also acts transitively on their images q_i, so this is equivalent to the stabilizer of e_i being contained in the stabilizer of q_i. Now U_d is abelian, so each q_i has the same stabilizer, namely $K = \ker\alpha = C_G(Q)$; since z generates the stabilizer of a face, we require that K contains z, and hence contains its normal closure Z in G. Putting $z = 1$ in the presentation for G shows that $G/Z \cong C_h$ where $h = \mathrm{hcf}(m, 2)$, so there is one K, with $G/K \cong C_k$, for each k dividing h. Thus $G\alpha$ is cyclic, generated by $u = x\alpha = (y\alpha)^{-1}$ satisfying $u^h = 1$, and $q_1, \dots, q_f$ have the form qu^i for some q. They are all associates in $\mathbf{Z}_d$, so θ is an epimorphism if and only if q is a unit. We then have $P_1 \le \ker\theta$ if and only if $0 = \left(\sum e_i\right)\theta = \sum q_i = q(1+u+u^2+\cdots+u^{f-1})$ in $\mathbf{Z}_d$, or equivalently, $1+u+u^2+\cdots+u^{f-1} = 0$. (Since u has order k, we can write this in the form $l(1+u+u^2+\cdots+u^{k-1}) = 0$ where $l = |K : \langle z\rangle| = f/k$.) Each solution u of $(*)$ gives a one-dimensional quotient of P/P_1, and hence of $\overline{M}$, so the corresponding subgroup N of M gives a d-sheeted regular cyclic covering $\mathcal{N}$ of $\mathcal{M}$. The action of G by conjugation on $D = \langle z^n\rangle$ is the same as its action on Q, so $\mathrm{Aut}\,\mathcal{N}$ has the stated presentation. □

As in the central case, $\mathcal{N}$ has type $\{dn, m\}$ and genus $(d-1)(f-2)/2$. Being regular, $\mathcal{N}$ is reflexible if and only if the corresponding subgroup N is invariant under the outer automorphism of Δ which inverts the generators x and y. Since $\mathcal{M}$ is reflexible, this happens if and only if $u = u^{-1}$; now $u^h = 1$ with h dividing 2, so the maps $\mathcal{N}$ are all reflexible.

The above arguments are all valid for $d = \infty$, with the usual interpretations such as $\mathbf{Z}_\infty = \mathbf{Z}$, so $U_\infty = \{\pm 1\}$. Unlike in the central case, however, condition $(*)$ does not imply that d must be finite: when m and f are even we could have $d = \infty$ with $u = -1$, giving an infinite-sheeted cyclic covering of $\mathcal{M}$ by a non-compact reflexible map $\mathcal{N}$ of type $\{\infty, m\}$. When d is finite, $\mathcal{N}$ has genus $(d-1)(f-2)/2$ as in the central case.

9. THE COVERINGS

We now consider all the normal subgroups K of the Platonic groups G containing z, describing the resulting cyclic coverings $\mathcal{N}$ in each case. When $m = 4$, so that $h = 2$, the coverings corresponding to $u = -1$ are described, using a slightly different construction, by Conder and Kulkarni in [3].

(a) If $k = 1$ then $K = G$ and we have the central coverings described earlier. In particular, if m is odd then $h = \mathrm{hcf}(m, 2) = 1$, so these are the only coverings; this applies to $\mathcal{M} = \{3, 3\}, \{4, 3\}, \{5, 3\}, \{3, 5\}$ and $\{2, m\}$ for m odd. We therefore assume that $k > 1$, so $u \ne 1$.

(b) If $\mathcal{M}$ is the octahedron $\{3, 4\}$, then $h = 2$ and G $(\cong S_4)$ has a normal subgroup $K = G' \cong A_4$ of index $k = 2$. This contains the stabilizer $\langle z\rangle$ in G of a face of $\mathcal{M}$, with index $l = 4$, so condition $(*)$ is $4(1+u) = 0$ where $u^2 = 1 \ne u$ in $\mathbf{Z}_d$.

When d is finite, the conditions $4(1+u) \equiv 0$ and $u^2 \equiv 1$ are satisfied mod (d) if and only if they are satisfied mod (p^e) for each prime-power p^e in the factorization of d. When p is odd there is a unique solution $u \equiv -1 \bmod (p^e)$. In $\mathbf{Z}_2$ there is a unique solution $u = 1$, in $\mathbf{Z}_4$ there are two solutions $u = \pm 1$, and in $\mathbf{Z}_8$ there are four solutions $u = \pm 1, \pm 3$; if $e \ge 4$ the solutions of $u^2 = 1$ in $\mathbf{Z}_{2^e}$ are $u = \pm 1, 2^{e-1} \pm 1$, and of these just $u = -1, 2^{e-1} - 1$ satisfy $4(1+u) = 0$. It follows that when $d = 2^e$ the number of solutions $u \in \mathbf{Z}_d$ of $4(1+u) = 0$ and $u^2 = 1 \ne u$ is equal to $0, 1, 3$ or 2 as $e \le 1, e = 2, e = 3$ or $e \ge 4$; when $d = 2^e d'$ with odd $d' > 1$ the number of solutions is $1, 2, 4$, or 2, respectively. This is therefore the number of non-central d-sheeted cyclic coverings $\mathcal{N}$ obtained by branching over the faces of

$\mathcal{M}$. They are formed by assigning the monodromy permutations 1 and u (in $\mathbf{Z}_d$) to the centers of alternate faces of $\mathcal{M}$. Since $k = 2$, each map $\mathcal{N}$ is reflexible; it has type $\{3d, 4\}$, with eight $3d$-gonal faces, and $6d$ vertices of valency 4, so its genus is $3(d-1)$. Thus for each g divisible by 3 there is a regular map of genus g with $8(g+3)$ automorphisms (see [1, 12] for the analogous result for Riemann surfaces). The group

$$\operatorname{Aut}\mathcal{N} = \langle x, y, z \mid x^4 = y^2 = z^{3d} = xyz = 1,\ (z^3)^x = z^{3u}\rangle$$

has a cyclic normal subgroup $D = \langle z^3\rangle$ of order d.

When $d = \infty$ there is a unique solution $u = -1$ of $(*)$, giving a non-compact reflexible map $\mathcal{N}$ of type $\{\infty, 4\}$; this is an infinite-sheeted cyclic covering of $\mathcal{M}$, with

$$\operatorname{Aut}\mathcal{N} = \langle x, y, z \mid x^4 = y^2 = xyz = 1,\ (z^3)^x = z^{-3}\rangle.$$

(c) If $\mathcal{M}$ is the dihedron $\{n, 2\}$, so $G \cong D_n$, then $h = 2$ and we can take $K = \langle z\rangle \cong C_n$, the subgroup stabilizing the two faces, with index $k = 2$. We have $l = 1$ and $f = 2$, so condition $(*)$ gives $1 + u = 0$ where $u^2 = 1 \neq u$ in $\mathbf{Z}_d$, that is, $u = -1$ and $d > 2$.

When $d = \infty$ the solution $u = -1$ gives a non-compact reflexible map $\mathcal{N}$ of type $\{\infty, 2\}$; this is an infinite-sheeted cyclic covering of $\mathcal{M}$, with

$$\begin{aligned}\operatorname{Aut}\mathcal{N} &= \langle x, y, z \mid x^2 = y^2 = xyz = 1,\ (z^n)^x = z^{-n}\rangle\\ &= \langle x, y \mid x^2 = y^2 = 1\rangle\\ &\cong D_\infty.\end{aligned}$$

One can realize $\mathcal{N}$ in the complex plane $\mathbf{C}$, with vertices at the integers, edges along the real line joining consecutive integers, and the upper and lower half-planes as the two faces. The function $z \mapsto e^{2\pi i z/n}$, which has period n, gives the projection of $\mathcal{N}$ onto the map $\mathcal{M}$ on Σ.

When d is finite $\mathcal{N}$ is the reflexible map $\{dn, 2\}$ of genus 0, with

$$\begin{aligned}\operatorname{Aut}\mathcal{N} &= \langle x, y, z \mid x^2 = y^2 = z^{dn} = xyz = 1,\ (z^n)^x = z^{-n}\rangle\\ &= \langle x, y, z \mid x^2 = y^2 = z^{dn} = xyz = 1\rangle\\ &\cong D_{dn}.\end{aligned}$$

(d) Let $\mathcal{M}$ be the hosohedron $\{2, m\}$, so $G \cong D_m$. If m is even then $h = 2$, so we can take $K = \langle x^2, z\rangle \cong D_l$, of index $k = 2$; this contains $\langle z\rangle$ with index $l = m/2$. The relevant conditions are therefore $l(1+u) = 0, u^2 = 1 \neq u$ in $\mathbf{Z}_d$. When $d = \infty$ the solution $u = -1$ gives a non-compact reflexible map $\mathcal{N}$ of type $\{\infty, m\}$; this is an infinite-sheeted cyclic covering of $\mathcal{M}$, with

$$\operatorname{Aut}\mathcal{N} = \langle x, y, z \mid x^m = y^2 = xyz = 1,\ (z^2)^x = z^{-2}\rangle.$$

Now let d be finite. For odd p, the conditions $l(1+u) \equiv 0,\ u^2 \equiv 1 \bmod (p^e)$ are satisfied by $u \equiv 1$ with $l \equiv 0$, and by $u \equiv -1$ for any l. For $p^e = 2$, the solutions are $u \equiv 1$ for any l; for $p^e = 4$ they are $u \equiv 1$ with l even, and $u \equiv -1$ for any l; for $p = 2$ and $e \geq 3$ they are $u \equiv 1$ or $2^{e-1} + 1$ with $2^{e-1} \mid l$, and $u \equiv -1$ for any l, and $u \equiv 2^{e-1} - 1$ with l even. If $d = 2^e p_1^{e_1} \dots p_r^{e_r}$, where $p_1, \dots, p_r$ are distinct odd primes and each $e_i \geq 1$, then the number of involutions $u \in \mathbf{Z}_d$ is

$$\nu = \begin{cases} 2^r - 1 & \text{if } e \leq 1,\\ 2^{r+1} - 1 & \text{if } e = 2,\\ 2^{r+2} - 1 & \text{if } e \geq 3,\end{cases}$$

and for each u the above conditions determine which values of l (and hence of $m = 2l$) correspond to coverings. For instance, l must be divisible by any odd $p_i^{e_i}$ such that $u \equiv 1 \bmod (p_i^{e_i})$; if $u = -1$ we obtain a covering for every l. These maps $\mathcal{N}$ are reflexible, of type $\{2d, m\}$ and genus $g = (l-1)(d-1)$, with

$$\mathrm{Aut}\,\mathcal{N} = \langle x, y, z \mid x^m = y^2 = z^{2d} = xyz = 1,\ (z^2)^x = z^{2u}\rangle.$$

Taking $m = 4$, so $l = 2$, the solution $u = -1$ gives a map of type $\{2d, 4\} = \{2g+2, 4\}$ and genus g with $8(g+1)$ automorphisms for each $g \geq 0$; by comparing their automorphism groups, one can show that this is isomorphic to Sherk's map $\{2d, 2 \cdot 2\}$, the dual of the map $\{2 \cdot 2, 2d\}$ constructed in Section 4(g) as a double covering of $\{2, 2d\}$.

10. Branching Over Vertices or Edges

Just as in the central case, we can also consider cyclic coverings branched over the vertices or the edges. The maps obtained by branching over the vertices are just the duals of those obtained by branching over the faces, so we will not discuss these. Branching over the edges, we require that the normal subgroup $K = \ker\alpha$ of G should contain the stabilizer $\langle y\rangle$ of an edge; there is one such subgroup of index k for each k dividing $h = \mathrm{hcf}(m, n)$. The value of l in condition $(*)$ is then given by

$$l = e/k = |K : \langle y\rangle|,$$

where $e = |G|/2$ is the number of edges of $\mathcal{M}$. The hypermap $\mathcal{N}$ has type $(m, 2d, n)$, with

$$\mathrm{Aut}\,\mathcal{N} = \langle x, y, z \mid x^m = y^{2d} = z^n = xyz = 1,\ (y^2)^x = y^{2u}\rangle,$$

where $u = x\alpha$ and $D = \langle y^2\rangle$. The coverings obtained for each $\mathcal{M}$ are as follows.

(a) If $k = 1$ then $K = G$ and we have the central coverings listed earlier. In particular, this is the only possibility if $h = \mathrm{hcf}(m, n) = 1$, so this deals with $\mathcal{M} = \{3, 4\}, \{4, 3\}, \{3, 5\}, \{5, 3\}$ and $\{2, m\}$, $\{n, 2\}$ for m, n odd.

(b) If $\mathcal{M}$ is the tetrahedron $\{3, 3\}$, with $G \cong A_4$, then $h = 3$ and we can take $K = G' \cong V_4$, of index $k = 3$. This contains the stabilizer $\langle y\rangle$ of an edge, with index $l = 2$, so condition $(*)$ is $2(1 + u + u^2) = 0$ where $u^3 = 1 \neq u$ in $\mathbf{Z}_d$. This implies that d is finite. The conditions $2(1+u+u^2) = 0$ and $u^3 = 1$ are satisfied mod (d) if and only if they are satisfied mod (p^e) for each prime-power p^e in the factorization of d. When $p = 2$ or 3 the unique solution $u = 1$ in $\mathbf{Z}_p$ does not lift to a solution in $\mathbf{Z}_{p^e}$ for any $e \geq 2$. When $p \equiv 1 \bmod (3)$ there are exactly two solutions in $\mathbf{Z}_{p^e}$, the elements of order 3 in the cyclic group U_{p^e}, and when $2 \neq p \equiv 2 \bmod (3)$ there are none. It follows that there are solutions in $\mathbf{Z}_d$ if and only if $d = 2^a 3^b p_1^{e_1} \dots p_r^{e_r}$ where $a, b \leq 1$ and $p_1, \dots, p_r$ are distinct primes $\equiv 1 \bmod (3)$, in which case there are 2^r solutions $u \neq 1$ if $r \geq 1$, and none if $r = 0$. Each solution gives a non-central d-sheeted cyclic covering $\mathcal{N}$, obtained by assigning the monodromy permutations $1, u$ and u^2 to the midpoints of pairs of opposite edges of $\mathcal{M}$. Each $\mathcal{N}$ is a hypermap of type $(3, 2d, 3)$, with $4d$ vertices, 6 edges and $4d$ faces, so its genus is $2(d-1)$, and

$$\mathrm{Aut}\,\mathcal{N} = \langle x, y, z \mid x^3 = y^{2d} = z^3 = xyz = 1,\ (y^2)^x = y^{2u}\rangle.$$

Since $k > 2$, $\mathcal{N}$ is not reflexible: the hypermaps corresponding to the units u and u^{-1} $(= u^2)$ form a chiral pair $\mathcal{N}$, $\overline{\mathcal{N}}$. Since inverting u is equivalent to transposing the generators x and z, $\overline{\mathcal{N}}$ is isomorphic to the vertex-face dual $\mathcal{N}'$ of $\mathcal{N}$.

(c) Let $\mathcal{M}$ be the hosohedron $\{2, m\}$, so $G \cong D_m$, and let m be even. Then $h = 2$ and the subgroup $K = \langle x^2, y\rangle \cong D_{m/2}$ of index $k = 2$ contains the stabilizer $\langle y\rangle$ of an edge. The hypermaps of type $(m, 2d, 2)$ obtained by branching over the edges of $\mathcal{M}$ are the edge-face duals of the maps of type $\{2d, m\}$, regarded as hypermaps of type $(m, 2, 2d)$, which we obtained earlier by branching over the faces of $\mathcal{M}$; thus there is nothing essentially new here.

(d) If $\mathcal{M}$ is the dihedron $\{n, 2\}$, so $G \cong D_n$, and if n is even, then $h = 2$ and we can take $K = \langle y, z^2\rangle \cong D_{n/2}$, with $k = 2$. The hypermaps of type $(2, 2d, n)$ obtained by branching over the edges of $\mathcal{M}$ are the vertex-face duals of those of type $(n, 2d, 2)$ obtained as in the previous example by branching over the edges of $\mathcal{M}' = \{2, n\}$, so again we find nothing new.

11. Coverings of the Star Maps

For completeness, we conclude by describing the cyclic coverings of the star maps $\mathcal{M} = \mathcal{S}_n$. These are reflexible hypermaps of type $(n, 1, n)$ and genus 0, consisting of a single vertex of valency n (the point $0 \in \Sigma$), n edges of valency 1 ('half-edges' in map terminology) given by $re^{2\pi ij/n}$ where $0 \le r \le 1$ and $j = 0, 1, \ldots, n-1$, and a single n-gonal face centered at ∞. The automorphism group is $G = \Delta(n, 1, n) \cong C_n$, generated by a rotation through $2\pi/n$ around 0 and ∞. These are the only regular hypermaps on the sphere, other than the Platonic maps described earlier and the hypermaps obtained from them by various dualities. Although these may appear rather degenerate objects, they can be very useful: for instance, many vertex-transitive maps arise naturally as coverings of star maps.

Non-trivial branching over the single vertex or face center of $\mathcal{S}_n$ is impossible, since it is easily seen that a covering of the sphere cannot have a single branch-point. If we allow branching over 0 and ∞, then we obtain the star map $\mathcal{N} = \mathcal{S}_{dn}$, a d-sheeted cyclic central covering for each d.

If we allow branching over the edge centers (the nth roots of 1 in Σ), then condition $(*)$ becomes $u^n = 1$ and $1 + u + u^2 + \cdots + u^{e-1} = 0$ in $\mathbf{Z}_d$; here e is the number n of edges of $\mathcal{S}_n$, so the second condition implies the first, and we therefore require simply

$$1 + u + u^2 + \cdots + u^{n-1} = 0. \qquad (**)$$

Each solution gives rise to a regular hypermap $\mathcal{N}$ of type (n, d, n) and genus $(d-1)(n-2)/2$, which is reflexible if and only if $u^2 = 1$. It has automorphism group

$$\begin{aligned}\tilde{G} &= \langle x, y, z \mid x^n = y^d = z^n = xyz = 1, y^x = y^u\rangle \\ &= \langle x, y \mid x^n = y^d = 1, y^x = y^u\rangle,\end{aligned}$$

since the relation $(xy)^n = 1$ is implied by the others together with $(**)$; this is a split extension of a normal subgroup $\langle y\rangle \cong C_d$ by a complement $\langle x\rangle \cong C_n$. Central coverings are given by solutions $u = 1$, with $\tilde{G} \cong D \times G \cong C_d \times C_n$, and these exist for each d dividing n.

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